Lecture 33: Higher Order Derivatives

33.1 Higher order derivatives

If f is differentiable, then its derivative f' may also be differentiable. The derivative of f' is called the *second derivative* of f and is denoted f''.

Example If $f(x) = x^3 - 6x^5$, then

$$f'(x) = 3x^2 - 30x^4$$

and

$$f''(x) = 6x - 120x^3.$$

Higher order derivatives are denoted similarly. For example, for the previous example,

$$f'''(x) = 6 - 360x^2$$

and

$$f''''(x) = -720x.$$

Note that we also let $f^{(4)}(x) = f''''(x)$.

Example Note that for the previous example,

$$f^{(5)}(x) = -720$$

and

$$f^{(6)}(x) = 0.$$

Moreover, note that

$$f^{(n)}(x) = 0$$

for any $n \geq 6$. For example,

$$f^{(100)}(x) = 0.$$

33.2 Leibniz notation

If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$, $\frac{d^2y}{dx^2} = f''(x)$, $\frac{d^3y}{dx^3} = f'''(x)$, etc.

Example If $y = \frac{1}{x}$, then

$$\frac{dy}{dx} = -\frac{1}{x^2},$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3},$$

and

$$\frac{d^3y}{dx^3} = -\frac{6}{x^4}.$$

33.3 Seeing patterns in iterations

Note that we can sometimes see a pattern in successive derivatives of a given function. For example, if $f(x) = x^7$, then

$$f^{(7)}(x) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7! = 5040.$$

Definition In n is a positive integer, then

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

is called n *factorial*.

Note: $1! = 1$, and, by convention, we let $0! = 1$.

Example $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Example Let $f(x) = \frac{1}{x}$. Then

$$f'(x) = -\frac{1}{x^2},$$

$$f''(x) = \frac{2}{x^3},$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4} = -\frac{3!}{x^4},$$

and

$$f^{(4)}(x) = \frac{4!}{x^5}.$$

Continuing in this pattern, we expect that

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}.$$

For example,

$$f^{(10)}(x) = \frac{(-1)^{10} 10!}{x^{11}} = \frac{3,628,800}{x^{11}}.$$