Chapter 15

Differentiation

This is the first of six chapters which examine the calculus and its application to business, economics, and other areas of problem solving. Two major areas of study within the calculus are differential calculus and integral calculus. Differential calculus focuses on rates of change in analyzing a situation. Graphically, differential calculus solves of following problem: Given a function whose graph is a smooth curve and given a point in the domain of the function, what is the slope of the line tangent to the curve at this point.

Limits

Two concepts which are important in the theory of differential and integral calculus are the limit of a function and continuity.

Limits of Functions

In the calculus there is often an interest in the limiting value of a function as the independent variable approaches some specific real number. This limiting value, when it exists, is called **limit**. The notation

$$\lim_{x \to a} f(x) = L$$

is used to express the limiting value of a function. When investigating a limit, one is asking whether $f(x)$ approaches a specific value L as the value of x gets closer and closer to a.

Notation of Limits

The notation

$$\lim_{x \to a^-} f(x) = L$$

represents the limit of $f(x)$ as x approaches 'a' from the left (left-hand limit).
The notation

$$\lim_{x \to a^+} f(x) = L$$

represents the limit of $f(x)$ as x approaches 'a' from the right (right-hand limit).
If the value of the function approaches the same number L as x approaches 'a' from either direction, then the limit is equal to L.

Test For Existence Of Limit

If $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$ then

$$\lim_{x \to a} f(x) = L$$

If the limiting values of $f(x)$ are different when x approaches a from each direction, then the function does not approach a limit as x approaches a.

Example
Determine whether the $\lim_{x \to 2} x^3$ exists or not?

Solution: In order to determine the limit

let's construct a table of assumed value for x and corresponding values for f(x).
The following table indicates these values.

Approaching x = 2 from the left

x	1	1.5	1.9	1.95	1.99	1.995	1.999
$f(x) = x^3$	1	3.375	6.858	7.415	7.881	7.94	7.988

Approaching x = 2 from the Right

x	3	2.5	2.1	2.05	2.01	2.005	2.001
$f(x) = x^3$	27	15.625	9.261	8.615	8.121	8.060	8.012

Note that the value of x=2 has been approached from both the left and the right. From either direction, f(x) is approaching the same value, 8.

Note: For more examples, consult lectures and book.

Properties Of Limits And Continuity

1. If $f(x) = c$, where c is real, then $\lim_{x \to a} f(x) = c$
2. If $f(x) = x^n$, where n is a positive integer, then
   $\lim_{x \to a} x^n = a^n$
3. If $f(x)$ has a limit as $x \to a$ and c is any real number, then

$$\lim_{x \to a} c.f(x) = c.\lim_{x \to a} f(x)$$

4. If $\lim_{x \to a} f(x)$and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

5. If $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$exist, then

$$\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$$

6. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

Limits and Infinity

Frequently there is an interest in the behavior of a function as the independent variable becomes large without limit ("approaching" either positive or negative infinity).

Horizontal Asymptote

The line y = a is a horizontal asymptote of the graph of f if and only if

$$\lim_{x \to \infty} f(x) = a$$

Note: See detailed discussion and examples in books.

Vertical Asymptote

The line x = a is a vertical asymptote of the graph of f if and only if

$$\lim_{x \to a} f(x) = \infty$$

<u>Continuity</u>

In an informal sense, a function is described as continuous if it can be sketched without lifting your pen or pencil from the paper (i.e., it has no gaps, no jumps, and no breaks). A function that is not continuous is termed as discontinuous.

Continuity at a Point

A function f is said to be continuous at x = a if

1- the function is defined at x = a, and
2- $\lim_{x \to a} f(x) = f(a)$

Examples
Determine that $f(x) = x^3$ is continuous at $x = 2$.
Solution: Since

$$\lim_{x \to 2} x^3 = f(2) = 8.$$

Thus f is continuous at $x = 2$.

Average Rate of Change and the Slope

The slope of a straight line can be determined by the two-point formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope provides an *exact measure* of the rate of change in the value of *y* with respect to a change in the value of *x.*

With nonlinear functions the rate of change in the value of y with respect to a change in x is not constant. One way of describing nonlinear functions is by the average rate of change over some interval.

In moving from point A to point B, the change in the value of x is $(x + \Delta x) - x$, or Δx. The associated change in the value of y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The ratio of these changes is.

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The above equation is sometimes referred to as the difference quotient.

<u>What Does The Difference Quotient Represent</u>

Given any two points on a function f having coordinates $[x, f(x)]$ and $[(x + \Delta x), f(x + \Delta x)]$, the difference quotient represents.

1. The average rate of change in the value of y with respect to the change in x while moving from

$$[x, f(x)] \ to \ [(x + \Delta x), f(x + \Delta x)]$$

2. The slope of the secant line connecting the two points.

The Derivative

DEFINITION:

Given a function of the form y = f(x), the derivative of the function is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If this limit exists.

Comments About the Derivative

i. The above equation is the general expression for the derivative of the function f.
ii. The derivative represents the instantaneous rate of change in the dependent variable given a change in the independent variable. The notation dy/dx is used to represent the instantaneous rate of change in y with respect to a change in x. This notation is distinguished from $\Delta y/\Delta x$ which represents the average rate of change.
iii. The derivative is a general expression for the slope of the graph of f at any point x in the domain.
iv. If the limit in the above figure does not exist, the derivative does not exist.

Finding The Derivative (Limit Approach)

Step 1 Determine the difference quotient for the given function.

Step 2 Find the limit of the difference quotient as Δx → 0.

Example

Find the derivative of $f(x) = -x^2$.

Solution: (In Lectures) .

Using And Interpreting The Derivative

To determine the instantaneous rate of change (or equivalently, the slope) at any point on the graph of a function f, substitute the value of the independent variable into the expression for $\frac{dy}{dx}$. The derivative, evaluated at $x = c$, can be denoted by ,

$$\frac{dy}{dx}|\_x = c,$$

which is read as "the derivative of y with respect to x evaluated at $x = c$".

Differentiation

The process of finding the derivative is called the differentiation.

A set of rules of differentiation exists for finding the derivative of many common functions. An alternate to $\frac{dy}{dx}$ notation is to let f' represent the derivative of the function f at x.

Rules of Differentiation

Rule 1: Constant Function

If $f(x) = c$, where c is any constant then

$$f'(x) = 0.$$

Rule 2: Power Rule

If $f(x) = x^n$, where n is a real number

$$f'(x) = nx^{n-1}$$

Algebra Flashback

Recall that $\frac{1}{x^n} = x^{-n}, \sqrt[n]{x^m} = x^{\frac{m}{n}}$.

Rule 3: Constant Times a Function

If $f(x) = c.g(x)$, where c is a constant and g is a differentiable function then

$$f'(x) = c.g'(x)$$

Rule 4: Sum or Difference of Functions

If $f(x) = u(x) \pm v(x)$, where u and v are differentiable,

$$f'(x) = u'(x) \pm v'(x),$$

Rule 5: Product Rule

If $f(x) = u(x).v(x)$, where u and v are differentiable,

$$f'(x) = u'(x).v(x) + v'(x).u(x)$$

Rule 6: Quotient Rule

If $f(x) = \frac{u(x)}{v(x)}$, where u and v are differentiable and $v(x) \neq 0$, then

$$f'(x) = \frac{v(x).u'(x) - u(x).v'(x)}{[v(x)]^2}$$

Rule 7: If $f(x) = [u(x)]^n$ then

$$f'(x) = n \cdot [u(x)]^{n-1} \cdot u'(x)$$

RULE 8: Base-e Exponential Functions

If $f(x) = e^{u(x)}$, where u is differentiable, then

$$f'(x) = u'(x)e^{u(x)}$$

Rule 9: Natural Logarithm Functions

If$f(x) = ln\, u(x)$, where u is differentiable, then

$$f'(x) = \frac{u'(x)}{u(x)}$$

Rule 10: Chain Rule

If $y = f(u)$, is a differentiable function and $u = g(x)$ is a differentiable function, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example:

An object is dropped from a cliff which is, 1,296 feet above the ground. The height of the object is described as a function of time. The function is

$$h = f(t) = -16t^2 + 1{,}296$$

where h equals the height in feet and *t* equals the time measured in seconds from the time the object is dropped.

a) How far will the object drop in 2 seconds?
b) What is the instantaneous velocity of the object at t = 2?
c) What is the velocity of the object at the instant it hits the ground?

Solution:

a) $\Delta h \;= f(2) - f(0)$

$$= [16(2)2 + 1{,}296] - [-16(0)2 + 1{,}296]$$
$$= (-64 + 1{,}296) - 1{,}296 = -64$$

Thus, the object drops 64 feet during the first 2 seconds.

b) Since $f'(x) = -32t$, the object will have a velocity equal to

$$f'(x) = -32(2)$$
$$= -64 feet/second$$

at t=2. the minus sign indicates the direction of velocity (down)

c) In order to determine the velocity of the object when it hits the ground, we must know when it will hit the ground. The object will hit the ground when $h = 0$, or when

$$-16t\hat{}2 + 1{,}296 = 0$$

and $t = \pm 9$.

Since a negative root is meaningless, we can conclude that the object will hit the ground after 9 seconds. The velocity at this time will be

$$f'(9) = 32(9) = 288 \text{ feet/second.}$$

Exponential Growth Processes

The exponential growth processes were introduced in Chap. 7.

The generalized exponential growth function was presented as,

$$V = f(t) = V_0 e^{kt}$$

We indicated that these processes are characterized by a constant percentage rate of growth. To verify this, let's find the derivative.

$$f'(t) = V_0(k)e^{kt}$$

which can be written as

$$f'(t) = kV_0 e^{kt}$$

The derivative represents the instantaneous rate of change in the value V with respect to a change in t.

The percentage rate of change would be found by the ratio.

$$\frac{Instantaneous\ rate\ of\ change}{Value\ of\ the\ function} = \frac{f'(t)}{f(t)}$$

For this function,

$$\frac{f'(t)}{f(t)} = \frac{kV_0e^{kt}}{V_0e^{kt}}$$

This confirms that for an exponential growth function of the form of Eq. (7.9), k represents the percentage rate of growth. Given that k is a constant, the percentage rate of growth is the same for all values of t.

The Second Derivative:

The second derivative f" of a function is the derivative of the first derivative. At x, it is denoted by either $\frac{d^2y}{dx^2}$ or $f"(x)$.

The Second Derivative:

As the first derivative is a measure of the instantaneous rate of change in the value of y with respect to a change in x, the second derivative is a measure of the instantaneous rate of change in the value of the first derivative with respect to the change in x. Described differently, the second derivative is a measure of the instantaneous rate of change in the slope with respect to a change in x.

Example

Consider the function $f(x) = -x^2$. The first and second derivatives of this function are

$$f'(x) = -2x \qquad f"(x) = -2$$

Definition: nth-order Derivative

The n-th order derivative of f, denoted by $f^n(x)$ is found by differentiating the derivative of order $n-1$. That is, at x,

$$f^{(n)}(x) = \frac{d}{dx}[f^{(n-1)}(x)]$$