**Chapter 4: Mathematical Functions**

A **function** is a mathematical rule that assigns to each input value one and only one output value.

**The Domain of a function** is the set consisting of all possible input values.

**The range of a function** is the set of all possible output values.

**Notation**

The assigning of output values to corresponding input values is often called as mapping. The notation

𝑓(𝑥)=𝑦

represents the mapping of the set of input values of 𝑥 into the set of output values 𝑦, using the mapping rule 𝑓.

The equation

𝑦 = 𝑓(𝑥)

denotes a functional relationship between the variables 𝑥 and 𝑦. Here 𝑥 means the input variable and 𝑦 means the output variable, i.e. the value of 𝑦 depends upon and uniquely determined by the values of 𝑥.

The input variable is called the independent variable and the output variable is called the dependent variable.

**Some examples**

1. The fare of taxi depends upon the distance and the day of the week.

2. The fee structure depends upon the program and the type of education (on campus/off campus) you are admit ting in.

3. The house prices depend on the location of the house.

**Note**: The variable 𝑥 is not always the independent variable, 𝑦 is not always the dependent variable and 𝑓 is not always the rule relating 𝑥 and 𝑦. Once the notation of function is clear then, from the given notation, we can easily identify the input variable, output variable and the rule relating them, for example 𝑢=𝑔(𝑣) has input variable 𝑣, output variable 𝑢 and 𝑔 is the rule relating 𝑢 and 𝑣.

**Example (Weekly Salary Function**)

A person gets a job as a salesperson and his salary depends upon the number of units he sells each week. Then, dependency of weekly salary on the units sold per week can be represented as

𝑦=𝑓(𝑥), where 𝑓 is the name of the salary function. Suppose your employer has given you the following equation for determining your weekly salary:

𝑦=100 𝑥+5000

Given any value of x will result in the value of y with respect to the function f.

If x=5, then y=100(5)+5000 = 5500. We write this as, f(5)=5500.

**Example**

Given the functional relationship

f(x)=5x−10,

Find i) f(0),

ii) f(−2)

iii) f(a+b).

**Solution:** As 𝑓(𝑥)=5𝑥−10, so i) 𝑓(0)=5(0)−10=−10

ii)𝑓(−2)=5(−2)−10=−20

iii)𝑓(𝑎+𝑏)=5𝑎+5𝑏−10.

**Domain and Range**

Recall that the set of all possible input values is called the domain of a function. Domain consists of all real values of the independent variable for which the dependent variable is defined and real.

**Example**

Determine the domain of the function 𝑓(𝑥)= .

**Solution:** 𝑓(𝑥) is undefined at 4−𝑥2 =0, which gives that given function is not defined at 𝑥=±2. Thus

Domain(f) ={𝑥|𝑥 is real and 𝑥≠ ±2}

**Restricted domain and range**

Up to now we have solved mathematically to find the domains of some types of functions. But for some real world problems, there may be more restriction on the domain e.g. in the weekly salary equation:

𝑦=100 𝑥+5000

Clearly, the number of units sold per week can not be negative. Also, they can not be in fractions, so the domain in this case will be all positive natural numbers {1,2,3,…}. Further, the employer can also put the condition on the maximum number of units sold per week. In this case, the domain will be defined as:

D={1,2,…,u}

where u is the maximum number of units sold.

**Multivariate Functions**

For many mathematical functions, the value of the dependent variable depends upon more than one independent variable.

A function which contain more than one independent variable are called multivariate function*.*

A function having two independent variables is called bivariate function.

They are denoted by 𝑧=𝑓(𝑥,𝑦), where 𝑥 and 𝑦 are the independent variables and 𝑧 is the dependent variable e.g. 𝑧=2𝑥+5𝑦.

In general the notation for a function 𝑓 where the value of dependent variable depends on the values of 𝑛 independent variables is 𝑧=𝑓(𝑥 ,⋯,𝑥 ).

For example,

𝑧=2𝑥 +5𝑥 +4𝑥 −4𝑥 +𝑥.

**Types of Functions**

**Constant Functions**

A constant function has the general form

𝑦=𝑓(𝑥)=𝑎0

Here, domain is the set of all real numbers and range is the single value 𝑎0 , e.g. 𝑓(𝑥)=20.

**Linear Functions**

A linear function has the general (slope-intercept) form

𝑦=𝑓(𝑥)=𝑎 𝑥+𝑎0

where 𝑎 is slope and 𝑎0 is 𝑦-intercept. For example, 𝑦=2𝑥+3 is represented by a straight line with slope 2 and y-intercept 3.

The weekly salary function is also an example of linear function.

**Quadratic Function**

A quadratic function has the general form

𝑦=𝑓(𝑥)=𝑎2𝑥2 +𝑎1𝑥+𝑎0

provided that 𝑎2≠ 0, e.g. 𝑦=2𝑥2 +3.

**Cubic Function**

A cubic function has the general form

𝑦=𝑓(𝑥)=𝑎3𝑥3 +𝑎2𝑥2 +𝑎1𝑥+𝑎0

provided that 𝑎3≠ 0, e.g. 𝑦=𝑓(𝑥)=𝑥3 −50𝑥2 +10𝑥−1.

**Polynomial Functions**

A polynomial function of degree 𝑛 has the general form

𝑦=𝑓(𝑥)=𝑎n𝑥n +⋯+𝑎1𝑥+𝑎0

Where 𝑎 ,⋯,𝑎 and 𝑎 are real constants such that 𝑎n ≠ 0.

All the previous types of functions are polynomial functions.

**Rational Functions**

A rational function has the general form

𝑦=𝑓(𝑥)=𝑔(𝑥)/h(𝑥)

Where 𝑔(𝑥) and h(𝑥) are both polynomial functions, e.g.

𝑦=𝑓(𝑥)=2𝑥/5𝑥3 −2𝑥+3.

**Composite functions**

A composite function exists when one function can be viewed as a function of the values of another function. If 𝑦=𝑔(𝑢) and 𝑢= h(𝑥) then composite function

𝑦=𝑓(𝑥)=𝑔( h(𝑥)).

Here 𝑥 must be in the domain of and (𝑥) must be in the domain of 𝑔. For example, if 𝑦=𝑔(𝑢)=3𝑢2 +4𝑢 and 𝑢= h(𝑥)=𝑥+8, then 𝑔(h (−2))=132.

**Graphical Representation of Functions**

The function of one or two variables (independent) can be represented graphically. The functions of one independent variable are graphed in two dimensions, 2-space. The functions in two independent variables are graphed in three dimension, 3-space.

**Method of graphing**

1) To graph a mathematical function, one can simply assign different values from the domain of the independent variable and compute the values of dependent variable.

2) Locate the resulting order pairs on the co-ordinate axes, the vertical axis (𝑦-axis) is used to denote the dependent variable and the horizontal axis (𝑥-axis) is used to denote the independent variable.

3) Connect all the points approximately.

**Vertical Line Test**

By definition of a function, to each element in the domain there should correspond only one element in the range. This allows a simple graphical check to determine whether a graph represents a function or not. If a vertical line is drawn through any value in the domain, it will intersect the graph of the function at one point only. If the vertical line intersects at more than one point then, the graph depicts a relation and not a function.